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H^1–PERTURBATIONS OF SMOOTH SOLUTIONS FOR A WEAKLY DISSIPATIVE HYPERELASTIC-ROD WAVE EQUATION

M. BENDAHMANE, G. M. COCLITE, AND K. H. KARLSEN

ABSTRACT. We consider a weakly dissipative hyperelastic-rod wave equation (or weakly dissipative Camassa–Holm equation) describing nonlinear dispersive dissipative waves in compressible hyperelastic rods. By fixed a smooth solution, we establish the existence of a strongly continuous semigroup of global weak solutions for any initial perturbation from $H^1(\mathbb{R})$. In particular, the supersonic solitary shock waves [8] are included in the analysis.

1. Introduction and Statement of Main Results

Consider the equation

$$\partial_t u - \partial_{txx}^3 u + 3u\partial_x u + \delta\partial_{xx}^2 u = \gamma\big(2\partial_x u\partial_{xx}^2 u + u\partial_{xxx}^3 u\big), \quad t > 0,\ x \in \mathbb{R}. \tag{1.1}$$

In the case $\gamma = 1$, $\delta = 0$ it is known as the *Camassa-Holm equation* and describes unidirectional shallow water waves above a flat bottom: u represents the fluid velocity [1, 12]. The Camassa–Holm equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) [11, 1] and is completely integrable [1]. From a mathematical point of view the Camassa–Holm equation is well studied, see [3] for a complete list of references. In particular, we recall that existence and uniqueness results for global weak solutions have been proved by Constantin and Escher [4], Constantin and Molinet [5], and Xin and Zhang [17, 18], see also Danchin [9, 10].

When $\delta = 0$, it is termed *hyperelastic-rod wave equation* and describes the finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The constant $\gamma > 0$ depends on the material constants and the prestress of the rod [6, 7, 8].

The additional weakly dissipative term $\delta\partial_{xx}^2 u$ is introduced in [15]. We coin (1.1) the *weakly dissipative hyperelastic-rod wave equation.*

In [3] the authors consider the case $\delta = 0$ and prove the global existence and wellposedness of solutions belonging to $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$. On the other hand in [8] it is showed that for $\delta = 0$ and any constants $0 < \gamma < 3$, $c > 0$ there exists a $\zeta \in \mathbb{R}$

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such that the following peakon like function is a traveling wave solution of (1.1)

$$U(t,x) = \frac{c}{2}\Big(1 - \frac{1}{\gamma}\Big) + \frac{c}{2}\Big(\frac{3}{\gamma} - 1\Big)e^{-|x-ct-\zeta|/\sqrt{\gamma}}, \tag{1.2}$$

called *supersonic solitary shock wave*. It is clear that the analysis in [3] does not cover this kind of solutions (that do not belong to $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$!).

In this paper we extend the result of [3] to cover also (1.2). Roughly speaking the idea is to look at (1.2) as a $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))-$perturbation of a constant state. Indeed we can decompose U in the following way

$$U = U_1 + U_2, \quad U_1 := \frac{c}{2}\Big(1 - \frac{1}{\gamma}\Big), \quad U_2(t,x) := \frac{c}{2}\Big(\frac{3}{\gamma} - 1\Big)e^{-|x-ct-\zeta|/\sqrt{\gamma}},$$

where U_1 is a classical solution to (1.1) and U_2 is a perturbation that lies in the space $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$.

To be more precise: let $\varphi = \varphi(t,x)$ be a solution of (1.1) such that

$$\varphi \in C^3([0,\infty)\times\mathbb{R}), \qquad \varphi,\, \partial_t\varphi,\, \partial_x\varphi,\, \partial^2_{tx}\varphi,\, \partial^2_{xx}\varphi,\, \partial^3_x\varphi \in L^\infty(\mathbb{R}_+\times\mathbb{R}), \tag{1.3}$$

(this is the case if φ is periodic or constant) and

$$v_0 \in H^1(\mathbb{R}), \qquad \gamma > 0, \qquad \delta \in \mathbb{R}. \tag{1.4}$$

We want to study the wellposedness of the Cauchy problem

$$\begin{cases} \partial_t u - \partial^3_{txx}u + 3u\partial_x u + \delta\partial^2_{xx}u = \gamma\big(2\partial_x u\partial^2_{xx}u + u\partial^3_{xxx}u\big), & t>0,\ x\in\mathbb{R},\\ u(0,x) = \varphi(0,x) + v_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.5}$$

Observe that, at least formally, (1.5) is equivalent to the elliptic-hyperbolic system

$$\begin{cases} \partial_t u + \gamma u\partial_x u + \partial_x P = 0, & t>0,\ x\in\mathbb{R},\\ -\partial^2_{xx}P + P = \dfrac{3-\gamma}{2}u^2 + \dfrac{\gamma}{2}\big(\partial_x u\big)^2 + \delta\partial_x u, & t>0,\ x\in\mathbb{R},\\ u(0,x) = \varphi(0,x) + v_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.6}$$

Motivated by this, we shall use the following definition of weak solution. Moreover, in the same spirit of [3, Definition 1.1] we define the admissible perturbations.

Definition 1.1. *We call $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ a weak solution of the Cauchy problem* (1.5) *if*

- (*i*) $u \in C([0,\infty)\times\mathbb{R})$*;*
- (*ii*) $u - \varphi \in L^\infty\big((0,T); H^1(\mathbb{R})\big)$, $T > 0$*;*
- (*iii*) *u satisfies* (1.6) *in the sense of distributions;*
- (*iv*) $u(0,x) = \varphi(0,x) + v_0(x)$*, for every* $x \in \mathbb{R}$.

If, in addition, for each $T > 0$ there exists a positive constant K_T depending only on $\|v_0\|_{H^1(\mathbb{R})}$, φ, γ, T, such that

$$\partial_x\big(u(t,x) - \varphi(t,x)\big) \le \frac{4}{\gamma t} + K_T, \qquad (t,x)\in(0,T)\times\mathbb{R}, \tag{1.7}$$

then we say that $u - \varphi$ is an admissible perturbation of (1.1).

Our results are collected in the following theorem:

Theorem 1.1. *There exists a strongly continuous semigroup of solutions associated to the Cauchy problem* (1.5)*. More precisely, there exists a map*

$$S : [0,\infty) \times (0,\infty) \times \mathbb{R} \times H^1(\mathbb{R}) \longrightarrow C([0,\infty)\times\mathbb{R}), \quad (t,\gamma,\delta,v_0) \longmapsto S_t(\gamma,\delta,v_0)(\cdot),$$

with the following properties:

(*i*) *for each $v_0 \in H^1(\mathbb{R})$, $\gamma > 0$, $\delta \in \mathbb{R}$ the map $u(t,x) = S_t(\gamma,\delta,v_0)(x)$ is a weak solution of* (1.5) *and $u-\varphi$ is an admissible perturbation of* (1.1)*;*

(*ii*) *it is stable with respect to the initial condition and the coefficient in the following sense, if*

$$v_{0,n} \longrightarrow v_0 \text{ in } H^1(\mathbb{R}), \qquad \gamma_n \longrightarrow \gamma, \ \delta_n \longrightarrow \delta \text{ in } \mathbb{R}, \tag{1.8}$$

then

$$S(\gamma_n,\delta_n,v_{0,n}) - \varphi \longrightarrow S(\gamma,\delta,v_0) - \varphi \text{ in } L^\infty([0,T];H^1(\mathbb{R})), \tag{1.9}$$

for every $\{v_{0,n}\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$, $\{\gamma_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $v_0 \in H^1(\mathbb{R})$, $\gamma > 0$, $\delta \in \mathbb{R}$, $T > 0$.

Moreover, the following statements hold:

(*iii*) *the estimate* (1.7) *is valid with*

$$\begin{aligned} K_T := & \frac{2}{\sqrt{\gamma}}\Bigg(\Big(\frac{\gamma}{\sqrt{2}}\|\partial^2_{xx}\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + |\gamma-3|\sqrt{2}\|\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\Big)e^{\rho T}\|v_0\|_{H^1(\mathbb{R})} \\ & + \frac{\max\{|3-\gamma|,2\gamma\} + |3-\gamma|}{4}e^{2\rho T}\|v_0\|^2_{H^1(\mathbb{R})} \\ & + \frac{5\gamma}{2}\|\partial_x\varphi\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + 2\frac{\delta^2}{\gamma}\Bigg)^{1/2}, \end{aligned} \tag{1.10}$$

$$\rho := \frac{3+\gamma}{2}\|\partial_x\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + \frac{\gamma}{2}\|\partial^3_{xxx}\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + |\delta|, \tag{1.11}$$

for $T > 0$;

(*iv*) *there results*

$$\partial_x S(\gamma,\delta,v_0) \in L^p_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}), \tag{1.12}$$

with $1 \le p < 3$.

Our argument is based on the analysis of the evolution of the perturbation

$$v := u - \varphi.$$

From (1.5) we get the following equation for v

$$\begin{cases} \partial_t v - \partial^3_{txx}v + 3v\partial_x v + 3\varphi\partial_x v + 3v\partial_x\varphi + \delta\partial^2_{xx}v \\ \qquad = \gamma\big(2\partial_x v\partial^2_{xx}v + v\partial^3_{xxx}v + 2\partial_x v\partial^2_{xx}\varphi + 2\partial_x\varphi\partial^2_{xx}v + v\partial^3_{xxx}\varphi + \varphi\partial^3_{xxx}v\big), \\ v(0,\cdot) = v_0, \end{cases} \tag{1.13}$$

that is formally equivalent to the elliptic-hyperbolic system

$$\begin{cases} \partial_t v + \gamma v\partial_x v + \gamma v\partial_x\varphi + \gamma\varphi\partial_x v + \partial_x P = 0, \\ -\partial^2_{xx}P + P = \dfrac{3-\gamma}{2}v^2 + \dfrac{\gamma}{2}\big(\partial_x v\big)^2 + (3-\gamma)\varphi v + \gamma\partial_x\varphi\partial_x v + \delta\partial_x v, \\ v(0,\cdot) = v_0. \end{cases} \tag{1.14}$$

Since the argument is very similar to the one in [3] we simply sketch it.

2. Viscous Approximations: Existence and A Priori Estimates

We prove existence of a weak solution to the Cauchy problem (1.13) (and equivalently to (1.5)) by proving compactness of a sequence of smooth solutions $\{v_\varepsilon\}_{\varepsilon>0}$ solving the following viscous problems (see [2]):

(2.1)
$$\begin{cases}\partial_t v_\varepsilon+\gamma v_\varepsilon\partial_x v_\varepsilon+\gamma v_\varepsilon\partial_x\varphi+\gamma\varphi\partial_x v_\varepsilon+\partial_x P_\varepsilon=\varepsilon\partial_{xx}^2 v_\varepsilon,\\ -\partial_{xx}^2 P_\varepsilon+P_\varepsilon=\dfrac{3-\gamma}{2}v_\varepsilon^2+\dfrac{\gamma}{2}\big(\partial_x v_\varepsilon\big)^2+(3-\gamma)\varphi v_\varepsilon+\gamma\partial_x\varphi\partial_x v_\varepsilon+\delta\partial_x v_\varepsilon,\\ v_\varepsilon(0,\cdot)=v_{\varepsilon,0},\end{cases}$$

that is equivalent to the following fourth order one

$$\begin{cases}\partial_t v_\varepsilon-\partial_{txx}^3 v_\varepsilon+3v_\varepsilon\partial_x v_\varepsilon+3\varphi\partial_x v_\varepsilon+3v_\varepsilon\partial_x\varphi+\delta\partial_{xx}^2 v_\varepsilon\\ \qquad\qquad=\gamma\big(2\partial_x v_\varepsilon\partial_{xx}^2 v_\varepsilon+v_\varepsilon\partial_{xxx}^3 v_\varepsilon+2\partial_x v_\varepsilon\partial_{xx}^2\varphi+2\partial_x\varphi\partial_{xx}^2 v_\varepsilon\big)\\ \qquad+\gamma\big(\varphi\partial_{xxx}^3 v_\epsilon+v_\varepsilon\partial_{xxx}^3\varphi\big)+\varepsilon\partial_{xx}^2 v_\varepsilon-\varepsilon\partial_{xxxx}^4 v_\varepsilon,\\ v_\varepsilon(0,\cdot)=v_{\varepsilon,0}.\end{cases}\tag{2.2}$$

Formally, sending $\varepsilon\to 0$ in (2.2), (2.1) yields (1.13), (1.14), respectively.

We shall assume that

$$v_{\varepsilon,0}\in H^2(\mathbb{R}),\quad \|v_{\varepsilon,0}\|_{H^1(\mathbb{R})}\le\|v_0\|_{H^1(\mathbb{R})},\varepsilon>0,\quad\text{and}\quad v_{\varepsilon,0}\longrightarrow v_0\text{in}H^1(\mathbb{R}).\tag{2.3}$$

The starting point of our analysis is the following wellposedness result for (2.1) (see [2, Theorem 2.3]).

Lemma 2.1. *Assume* (1.3), (1.4) *and* (2.3), *let* $\varepsilon>0$. *There exists a unique smooth solution* $v_\varepsilon\in C\big([0,\infty);H^2(\mathbb{R})\big)$ *to the Cauchy problem* (2.1).

The next step in our analysis is to derive the following a priori estimates:

Lemma 2.2. *Assume* (1.3), (1.4) *and* (2.3), *and let* $\varepsilon>0$. *Then the following estimates hold:*

j) (Energy Conservation) for each $t\ge 0$

$$\big\|v_\varepsilon(t,\cdot)\big\|_{H^1(\mathbb{R})}^2+2\varepsilon\int_0^t\big\|\partial_x v_\varepsilon(\tau,\cdot)\big\|_{H^1(\mathbb{R})}^2\,d\tau\le e^{2\rho t}\|v_0\|_{H^1(\mathbb{R})}^2;\tag{2.4}$$

jj) (Oleinik type Estimate) for any $0<t<T$ *and* $x\in\mathbb{R}$,

$$\partial_x v_\varepsilon(t,x)\le\frac{4}{\gamma t}+K_T,\tag{2.5}$$

where K_T *is defined in* (1.10);

jj) (Higher Integrability Estimate) for every $0\le\alpha<1$, $T>0$, *and* $a,b\in\mathbb{R}$, $a<b$, *there exists a positive constant* C_T *depending only on* $\|v_0\|_{H^1(\mathbb{R})}$, φ, α, T, a *and* b, *but independent on* ε, *such that*

$$\int_0^T\int_a^b\big|\partial_x v_\varepsilon(t,x)\big|^{2+\alpha}\,dtdx\le C_T.\tag{2.6}$$

Remark 2.1. *Due to* [13, Theorem 8.5], (2.3) *and* (2.4), *we have for each* $t\ge 0$

$$\|v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\le\frac{1}{\sqrt{2}}\,\|v_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}\le\frac{e^{\rho t}}{\sqrt{2}}\|v_0\|_{H^1(\mathbb{R})}.\tag{2.7}$$

Proof of Lemma 2.2. We begin with j). Multiplying (2.2) by v_ε, integrating on $\mathbb{R}$, and integrating by parts we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(v_\varepsilon^2+(\partial_x v_\varepsilon)^2\right)dx+\varepsilon\int_{\mathbb{R}}\left((\partial_x v_\varepsilon)^2+(\partial_{xx}^2 v_\varepsilon)^2\right)dx\le\rho\int_{\mathbb{R}}\left(v_\varepsilon^2+(\partial_x v_\varepsilon)^2\right)dx,$$

where ρ is defined in (1.11). Hence (2.4) is consequence of (2.3) and the Gronwall Lemma.

We continue by proving jj). Introduce the notation

$$q_\varepsilon:=\partial_x v_\varepsilon.$$

From (2.1) we get the following equation for q_ε

$$\begin{aligned}\partial_t q_\varepsilon+\frac{\gamma}{2}q_\varepsilon^2+\gamma v_\varepsilon\partial_x q_\varepsilon+\gamma v_\varepsilon\partial_{xx}^2\varphi+\gamma\partial_x\varphi q_\varepsilon-\delta q_\varepsilon&\\+\gamma\varphi\partial_x q_\varepsilon+\frac{\gamma-3}{2}v_\varepsilon^2+(\gamma-3)\varphi v_\varepsilon+P_\varepsilon-\varepsilon\partial_{xx}^2 q_\varepsilon&=0.\end{aligned}\tag{2.8}$$

Using the fact that $e^{-|x|}/2$ is the Green's function of the operator $1-\partial_{xx}^2$

$$\begin{aligned}P_\varepsilon(t,x)=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-y|}\Big(&\frac{3-\gamma}{2}v_\varepsilon^2(t,y)+\frac{\gamma}{2}\big(\partial_x v_\varepsilon(t,y)\big)^2\\&+(3-\gamma)\varphi(t,y)v_\varepsilon(t,y)+\gamma\partial_x\varphi(t,y)\partial_x v_\varepsilon(t,y)\Big)\,dy.\end{aligned}\tag{2.9}$$

It follows from (2.4) and (2.7) (see [3, Proof of Lemma 3.1]) that

$$\left\|\gamma v_\varepsilon\partial_{xx}^2\varphi+\frac{\gamma-3}{2}v_\varepsilon^2+(3-\gamma)\varphi v_\varepsilon+P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\le L_T,\tag{2.10}$$

for some constant $L_T>0$. Then, from (2.8),

$$\partial_t q_\varepsilon+\frac{\gamma}{2}q_\varepsilon^2+\gamma v_\varepsilon\partial_x q_\varepsilon+\gamma\partial_x\varphi q_\varepsilon-\delta q_\varepsilon+\gamma\varphi\partial_x q_\varepsilon-\varepsilon\partial_{xx}^2 q_\varepsilon\le L_T.$$

Since

$$\frac{\gamma}{2}\xi^2+(\gamma\partial_x\varphi-\delta)\xi\ge\frac{\gamma}{4}\xi^2-\frac{(\gamma\partial_x\varphi-\delta)^2}{\gamma},\qquad\xi\in\mathbb{R},$$

we conclude

$$\begin{aligned}\partial_t q_\varepsilon+\frac{\gamma}{4}q_\varepsilon^2+\gamma v_\varepsilon\partial_x q_\varepsilon+\gamma\varphi\partial_x q_\varepsilon-\varepsilon\partial_{xx}^2 q_\varepsilon&\\\le L_T+2\gamma\|\partial_x\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}^2+2\frac{\delta^2}{\gamma}&=:\widetilde{L}_T.\end{aligned}\tag{2.11}$$

Employing the comparison principle for parabolic equations, we get

$$q_\varepsilon(t,x)\le h(t),\qquad 0\le t\le T,\ x\in\mathbb{R}\tag{2.12}$$

where h solves

$$\frac{dh}{dt}+\frac{\gamma}{4}h^2=\widetilde{L}_T,\qquad h(0)=\|\partial_x v_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}.\tag{2.13}$$

Since the map

$$H(t):=\frac{4}{\gamma t}+\sqrt{\frac{4\widetilde{L}_T}{\gamma}},\qquad t>0,$$

is a super-solution of (2.13) in the interval $[0,T]$. Due to the comparison principle for ordinary differential equations, we get $h(t)\le H(t)$ for all $0<t\le T$. Therefore, (2.5) is proved.

Finally, we consider jjj). The argument is very similar to the one of [3, Lemma 4.1]. Pick a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that

$$0 \le \chi \le 1, \qquad \chi(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \in (-\infty, a-1] \cup [b+1, \infty), \end{cases}$$

consider the map $\theta(\xi) := \xi\big(|\xi|+1\big)^\alpha$, $\xi \in \mathbb{R}$, then multiply (2.8) by $\chi\theta'(q_\varepsilon)$, integrate over $(0,T) \times \mathbb{R}$ and use (2.4). □

3. Compactness

Lemma 3.1. *The family $\{P_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty([0,T); W^{1,\infty}(\mathbb{R}))$ and $L^\infty([0,T); H^1(\mathbb{R}))$ for each $T > 0$.*

Proof. The argument is the same of [3, Lemma 5.1]: use the integral representation of P_ε (2.9) and then employ (2.7). □

Lemma 3.2. *There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and a function $v \in L^\infty([0,T); H^1(\mathbb{R})) \cap H^1([0,T] \times \mathbb{R})$, for each $T \ge 0$, such that*

$$v_{\varepsilon_j} \rightharpoonup v \quad \text{weakly in } H^1_{\mathrm{loc}}([0,T] \times \mathbb{R}), \text{ for each } T \ge 0, \tag{3.1}$$

$$v_{\varepsilon_j} \to v \quad \text{strongly in } L^\infty_{\mathrm{loc}}([0,\infty) \times \mathbb{R}). \tag{3.2}$$

Proof. Fix $T > 0$. Observe that, from (2.1),

$$\partial_t v_\varepsilon = \varepsilon \partial^2_{xx} v_\varepsilon - \gamma v_\varepsilon \partial_x v_\varepsilon - \gamma v_\varepsilon \partial_x \varphi - \gamma \varphi \partial_x v_\varepsilon - \partial_x P_\varepsilon,$$

hence, by (2.7), (2.4), Lemma 3.1, and the Hölder inequality, $\{v_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H^1([0,T] \times \mathbb{R}) \cap L^\infty([0,T); H^1(\mathbb{R}))$, and (3.1) follows. Finally, since $H^1(\mathbb{R}) \subset\subset L^\infty_{\mathrm{loc}}(\mathbb{R}) \subset L^2_{\mathrm{loc}}(\mathbb{R})$, (3.2) is consequence of [16, Theorem 5]. □

Lemma 3.3. *The family $\{P_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $W^{1,1}_{\mathrm{loc}}([0,T) \times \mathbb{R})$ for any $T > 0$. In particular, there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and a function $P \in L^\infty([0,T); W^{1,\infty}(\mathbb{R}))$ such that for each $1 \le p < \infty$*

$$P_{\varepsilon_j} \to P \quad \text{strongly in } L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}). \tag{3.3}$$

Proof. The argument is analogous to the one of [3, Lemma 5.3]. Using the integral representation (2.9) of P_ε and then employing (2.7) we get the uniform boundedness of $\big\{\partial_t P_\varepsilon\big\}_{\varepsilon>0}$ in $L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R})$. Then, due to Lemma 3.1, $\{P_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{1,1}_{\mathrm{loc}}([0,T) \times \mathbb{R})$. Finally, using again Lemma 3.1, we have the existence of a pointwise converging subsequence that is uniformly bounded in $L^\infty([0,T) \times \mathbb{R})$. Clearly, this implies (3.3). □

Lemma 3.4. *There exist a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and two functions $q \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R})$, $\overline{q^2} \in L^r_{\mathrm{loc}}([0,\infty) \times \mathbb{R})$ such that*

$$q_{\varepsilon_j} \rightharpoonup q \quad \text{in } L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}), \qquad q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad \text{in } L^\infty_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R})), \tag{3.4}$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in } L^r_{\mathrm{loc}}([0,\infty) \times \mathbb{R}), \tag{3.5}$$

for each $1 < p < 3$ and $1 < r < 3/2$. Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x) \quad \text{for almost every } (t,x) \in [0,\infty) \times \mathbb{R} \tag{3.6}$$

and

$$\partial_x v = q \quad \text{in the sense of distributions on } [0,\infty) \times \mathbb{R}. \tag{3.7}$$

Proof. Formulas (3.4) and (3.5) are direct consequences of Lemma 2.1 and (2.6). Inequality (3.6) is true thanks to the weak convergence in (3.5). Finally, (3.7) is a consequence of the definition of q_ε, Lemma 3.2, and (3.4). $\square$

In the following, for notational convenience, we replace the sequences $\{v_{\varepsilon_j}\}_{j\in\mathbb{N}}$, $\{q_{\varepsilon_j}\}_{j\in\mathbb{N}}$, $\{P_{\varepsilon_j}\}_{j\in\mathbb{N}}$ by $\{v_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$, $\{P_\varepsilon\}_{\varepsilon>0}$, respectively.

In view of (3.4), we conclude that for any $\eta \in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$ and any $1 \le p < 3$ we have

$$\eta(q_\varepsilon) \rightharpoonup \overline{\eta(q)} \quad \text{in } L^p_{\text{loc}}([0,\infty)\times\mathbb{R}), \qquad \eta(q_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\eta(q)} \quad \text{in } L^\infty_{\text{loc}}([0,\infty); L^2(\mathbb{R})). \tag{3.8}$$

Multiplying the equation in (2.8) by $\eta'(q_\varepsilon)$, we get

$$\begin{aligned} &\partial_t \eta(q_\varepsilon) + \gamma\partial_x\big((v_\epsilon+\varphi)\eta(q_\varepsilon)\big) - \varepsilon\partial^2_{xx}\eta(q_\varepsilon) - \varepsilon\eta''(q_\varepsilon)(\partial_x\eta(q_\varepsilon))^2 - \gamma q_\varepsilon\eta(q_\varepsilon) \\ &\quad + \gamma v_\varepsilon\partial^2_{xx}\varphi\eta'(q_\varepsilon) + \gamma\partial_x\varphi\big(q_\varepsilon\eta'(q_\varepsilon) - \eta(q_\varepsilon)\big) - \delta q_\varepsilon\eta'(q_\varepsilon) \\ &\quad + \frac{\gamma-3}{2}v_\varepsilon^2\eta'(q_\varepsilon) + \frac{\gamma}{2}q_\varepsilon^2\eta'(q_\varepsilon) + (\gamma-3)\varphi v_\varepsilon\eta'(q_\varepsilon) + P_\varepsilon\eta'(q_\varepsilon) = 0. \end{aligned} \tag{3.9}$$

Lemma 3.5. *For any convex $\eta \in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$, we have*

$$\begin{aligned} &\partial_t\overline{\eta(q)} + \gamma\partial_x\big((v+\varphi)\overline{\eta(q)}\big) - \gamma\overline{q\eta(q)} \\ &\quad + \gamma v\partial^2_{xx}\varphi\overline{\eta'(q)} + \gamma\partial_x\varphi\big(\overline{q\eta'(q)} - \overline{\eta(q)}\big) - \delta\overline{q\eta'(q)} \\ &\quad + \frac{\gamma-3}{2}v^2\overline{\eta'(q)} + \frac{\gamma}{2}\overline{q^2\eta'(q)} + (\gamma-3)\varphi v\overline{\eta'(q)} + P\overline{\eta'(q)} \le 0, \end{aligned} \tag{3.10}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$. Here $\overline{q\eta(q)}$, $\overline{q^2\eta'(q)}$ and $\overline{\eta'(q)q}$ denote the weak limits of $q_\varepsilon\eta(q_\varepsilon)$, $q_\varepsilon^2\eta'(q_\varepsilon)$ and $\eta'(q_\varepsilon)q_\varepsilon$ in $L^r_{\text{loc}}([0,\infty)\times\mathbb{R})$, $1<r<3/2$, respectively.

Proof. In (3.9), by convexity of η, (3.2), (3.4), and (3.5), sending $\varepsilon \to 0$ yields (3.10). $\square$

Remark 3.1. *From* (3.4) *and* (3.5), *it is clear that*

$$q = q_+ + q_- = \overline{q_+} + \overline{q_-}, \qquad q^2 = (q_+)^2 + (q_-)^2, \qquad \overline{q^2} = \overline{(q_+)^2} + \overline{(q_-)^2},$$

almost everywhere in $[0,\infty)\times\mathbb{R}$, where $\xi_+ := \xi\chi_{[0,+\infty)}(\xi)$, $\xi_- := \xi\chi_{(-\infty,0]}(\xi)$, $\xi\in\mathbb{R}$. Moreover, by (2.5) *and* (3.4),

$$q_\varepsilon(t,x), q(t,x) \le \frac{4}{\gamma t} + K_T, \qquad 0<t<T,\ x\in\mathbb{R}. \tag{3.11}$$

Lemma 3.6. *There holds*

$$\partial_t q + \gamma\partial_x\big((v+\varphi)q\big) - \frac{\gamma}{2}\overline{q^2} + \gamma v\partial^2_{xx}\varphi + \frac{\gamma-3}{2}v^2 + (\gamma-3)\varphi v + P - \delta q = 0, \tag{3.12}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Proof. Using (3.2), (3.3), (3.4), and (3.5), the result (3.12) follows by $\varepsilon \to 0$ in (2.8). $\square$

The next lemma contains a renormalized formulation of (3.12).

Lemma 3.7 ([3, Lemma 5.8]). *For any $\eta \in C^1(\mathbb{R})$ with $\eta' \in L^\infty(\mathbb{R})$,*

$$\begin{aligned}(3.13)\quad \partial_t \eta(q) + \gamma \partial_x\big((v+\varphi)\eta(q)\big) - \gamma q\eta(q) - \gamma\Big(\frac{\overline{q^2}}{2} - q^2\Big)\eta'(q)\\ + \gamma v \partial_{xx}^2 \varphi \eta'(q) + \gamma \partial_x \varphi\big(q\eta'(q) - \eta(q)\big) - \delta q \eta(q)\\ + \frac{\gamma-3}{2} v^2 \eta'(q) + (\gamma-3)\varphi v \eta'(q) + P\eta'(q) = 0,\end{aligned}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Following [3, Section 6] and [17], we improve the weak convergence of q_ε in (3.4) to strong convergence (and then we have an existence result for (1.5)). The idea is to derive a "transport equation" for the evolution of the defect measure $\big(\overline{q^2} - q^2\big)(t,\cdot) \geq 0$, so that if it is zero initially then it will continue to be zero at all later times $t > 0$. The proof is complicated by the fact that we do not have a uniform bound on q_ε from below but merely (3.11) and that in (2.6) we have only $\alpha < 1$.

Lemma 3.8. *Assume* (1.3) *and* (2.3). *Then for each $t \geq 0$*

$$\begin{aligned}(3.14)\quad &\int_{\mathbb{R}} \Big(\overline{(q_+)^2}(t,x) - (q_+)^2(t,x)\Big)\,dx\\ &\qquad \leq 2e^{\lambda t}\int_0^t \int_{\mathbb{R}} e^{-\lambda s} S(s,x)\,[\overline{q_+}(s,x) - q_+(s,x)]\,dsdx,\end{aligned}$$

where

$$\begin{aligned}\lambda &:= \gamma \|\partial_x\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + 2|\delta|,\\ S(s,x) &:= -v(s,x)\partial_{xx}^2\varphi(s,x) + \frac{3-\gamma}{2}v^2(s,x) + (3-\gamma)\varphi(s,x)v(s,x) - P(s,x).\end{aligned}$$

Proof. Let $T > 0$, $R > K_T$ (see (2.5)). Subtract (3.13) from (3.10) using the renormalization

$$\eta_R^+(\xi) := \begin{cases} R\xi - R^2/2, & \text{if } \xi > R,\\ \xi^2/2, & \text{if } 0 \leq \xi \leq R,\\ 0, & \text{if } \xi < 0.\end{cases}$$

Arguing as in [3, Lemma 6.4] we get

$$\frac{d}{dt}\int_{\mathbb{R}} \Big(\overline{(q_+)^2} - (q_+)^2\Big)\,dx \leq \lambda \int_{\mathbb{R}} \Big(\overline{(q_+)^2} - (q_+)^2\Big)\,dx + 2\int_{\mathbb{R}} S(t,x)\,[\overline{q_+} - q_+]\,dx,$$

for $4/(\gamma(R-K_T)) < t < T$. First we have to apply the Gronwall Lemma to the previous inequality on the interval $(4/(\gamma(R-K_T)), T)$. Then sending $R\to\infty$ and using (see [3, Lemma 6.2])

$$(3.15)\quad \lim_{t\to 0+}\int_{\mathbb{R}} \Big(\overline{\eta_R^+(q)}(t,x) - \eta_R^+(q(t,x))\Big)\,dx = 0, \qquad R > 0.$$

$\square$

Lemma 3.9. *For any $t \geq 0$ and any $R > 0$,*

$$\begin{aligned}(3.16)\quad &\int_{\mathbb{R}} \Big[\overline{\eta_R^-(q)}(t,x) - \eta_R^-(q)(t,x)\Big]\,dx\\ &\qquad \leq \frac{Re^{\lambda t}}{2}\int_0^t \int_{\mathbb{R}} e^{-\lambda s}\,(\gamma R - 2\gamma\partial_x\varphi + 2\delta)\,\overline{(R+q)\chi_{(-\infty,-R)}(q)}\,dsdx\end{aligned}$$

$$
\begin{aligned}
&- \frac{Re^{\lambda t}}{2} \int_0^t \int_{\mathbb{R}} e^{-\lambda s} \left(\gamma R - 2\gamma \partial_x \varphi + 2\delta\right)(R+q)\chi_{(-\infty,-R)}(q)\, ds dx \\
&+ \frac{\gamma R e^{\lambda t}}{2} \int_0^t \int_{\mathbb{R}} e^{-\lambda s} \left(\overline{(q_+)^2} - (q_+)^2\right) ds dx \\
&+ \gamma R e^{\lambda t} \int_0^t \int_{\mathbb{R}} e^{-\lambda s} \left(\overline{\eta_R^-(q)} - \eta_R^-(q)\right) ds dx \\
&+ e^{\lambda t} \int_0^t \int_{\mathbb{R}} e^{-\lambda s} S(s,x) \left[\overline{(\eta_R^-)'(q)} - (\eta_R^-)'(q)\right] ds dx.
\end{aligned}
$$

Proof. The argument is very similar to the one of [3, Lemma 6.3]. We begin by subtracting (3.13) from (3.10), using the renormalization

$$
\eta_R^-(\xi) := \begin{cases} 0, & \text{if } \xi > 0, \\ \xi^2/2, & \text{if } -R \le \xi \le 0, \\ -R\xi - R^2/2, & \text{if } \xi < -R. \end{cases}
$$

Then we integrate on $\mathbb{R}$ and use the Gronwall Lemma and (see [3, Lemma 6.2])

$$
\lim_{t \to 0+} \int_{\mathbb{R}} \left(\overline{\eta_R^-(q)}(t,x) - \eta_R^-(q(t,x))\right) dx = 0, \qquad R > 0. \tag{3.17}
$$

□

Lemma 3.10. *There holds* $\overline{q^2} = q^2$ *almost everywhere in* $[0,\infty) \times \mathbb{R}$.

Proof. We follow the argument of [3, Lemma 6.6]. We add (3.14) and (3.16). Using the concavity of $\xi \mapsto (R+\xi)\chi_{(-\infty,-R)}(\xi)$, the Gronwall Lemma, (3.15), (3.17) and

$$
\lim_{t \to 0+} \int_{\mathbb{R}} q^2(t,x)\, dx = \lim_{t \to 0+} \int_{\mathbb{R}} \overline{q^2}(t,x)\, dx = \int_{\mathbb{R}} \left(\partial_x v_0\right)^2 dx,
$$

we conclude that

$$
\int_{\mathbb{R}} \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_R^-(q)} - \eta_R^-(q)\right]\right)(t,x)\, dx = 0, \qquad \text{for each } 0 < t < T.
$$

By the Fatou Lemma, Remark 3.1, and (3.6), sending $R \to \infty$ yields

$$
0 \le \int_{\mathbb{R}} \left(\overline{q^2} - q^2\right)(t,x)\, dx \le 0, \qquad 0 < t < T,
$$

and, since the argument holds for each $T > 0$, we are done. □

4. Proof of Theorem 1.1

In this last section we prove Theorem 1.1. The first step consists in the proof of the existence of solutions for (1.5).

Lemma 4.1. *Assume* (1.3) *and* (2.3). *Then there exists an admissible weak solution of* (1.5), *satisfying* (*iii*) *and* (*iv*) *of Theorem 1.1.*

Proof. The conditions (*i*), (*ii*), (*iv*) of Definition 1.1 are satisfied, due to (2.3), (2.4) and Lemma 3.2. We have to verify (*ii*). Due to Lemma 3.10, we have

$$
q_\varepsilon \to q \qquad \text{strongly in } L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{R}). \tag{4.1}
$$

Clearly (3.2), (3.3), and (4.1) imply that v is a distributional solution of (1.14). Therefore $u := v + \varphi$ is a weak solution of (1.5) and v is an admissible perturbation of (1.1). Finally, (iii) and (iv) of Theorem 1.1 are consequences of (2.5) and (2.6), respectively. □

The second step is the existence of the semigroup.

Lemma 4.2. *There exists a strongly continuous semigroup of solutions associated with the Cauchy problem* (1.5)

$$S:\ [0,\infty)\times(0,\infty)\times\mathbb{R}\times H^1(\mathbb{R})\longrightarrow C([0,\infty)\times\mathbb{R}),$$

namely, for each $v_0\in H^1(\mathbb{R})$, $\gamma>0$, $\delta\in\mathbb{R}$ the map $u(t,x)=S_t(\gamma,\delta,v_0)(x)$ is an admissible weak solution and $u-\varphi$ and admissible perturbation of (1.5)*. Moreover,* (iii) *and* (iv) *of Theorem 1.1 are satisfied.*

Clearly, this lemma is a direct consequence of the following one and of the ones in the previous sections.

Lemma 4.3. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}\subset(0,\infty)$ and v, $w\in L^\infty([0,T];H^1(\mathbb{R}))\cap H^1([0,T]\times\mathbb{R})$, for each $T\geq 0$, be such that ε_n, $\mu_n\to 0$ and*

$$v_{\varepsilon_n}\to v,\quad v_{\mu_n}\to v,\quad \textit{strongly in } L^\infty([0,T];H^1(\mathbb{R})),\quad T>0,$$

then

$$v=w.$$

Proof. Let $t>0$. From [2, Theorem 3.1], we have that

$$\|v_\varepsilon(t,\cdot)-v_\mu(t,\cdot)\|_{H^1(\mathbb{R})}\leq A(t,\varepsilon+\mu)\|v_{0,\varepsilon}-v_{0,\mu}\|_{H^1(\mathbb{R})}+B(t,\varepsilon+\mu)|\varepsilon-\mu|,$$

with

$$A(t,\varepsilon+\mu)=\mathcal{O}\big(e^{t/(\varepsilon+\mu)}\big),\qquad B(t,\varepsilon+\mu)=\mathcal{O}\big(e^{t/(\varepsilon+\mu)}\big),$$

for each ε, $\mu>0$. Hence

$$\|v_{\varepsilon_n}(t,\cdot)-v_{\mu_n}(t,\cdot)\|_{H^1(\mathbb{R})}\leq c_1e^{t/(\varepsilon+\mu)}\big(|\varepsilon_n-\mu_n|+\|v_{0,\varepsilon_n}-v_{0,\mu_n}\|_{H^1(\mathbb{R})}\big),\quad n\in\mathbb{N},$$

for some constant $c_1>0$. Choosing suitable subsequences as in [3, Lemma 7.2] we get $v=w$. □

The third and last step is the stability of the semigroup.

Lemma 4.4. *The semigroup S defined on $[0,\infty)\times(0,\infty)\times\mathbb{R}\times H^1(\mathbb{R})$ satisfies the stability property* (ii) *of Theorem 1.1.*

Proof. Fix $\varepsilon>0$ and denote by S^ε the semigroup associated to the viscous problem (2.1) and $\widetilde{S}:=S-\varphi$. Choose $\{v_{0,n}\}_{n\in\mathbb{N}}\subset H^1(\mathbb{R})$, $\{\gamma_n\}_{n\in\mathbb{N}}\subset(0,\infty)$, $\{\delta_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$, $v_0\in H^1(\mathbb{R})$, $\gamma>0$, $\delta\in\mathbb{R}$ satisfying (1.8). The initial data satisfy $v_{0,\varepsilon,n}$, $v_{0,\varepsilon}\in H^2(\mathbb{R})$ and (2.3). Finally, write

$$v_{\varepsilon,n}:=S^\varepsilon(\gamma_n,\delta_n,v_{0,n}),\qquad v_n:=S(\gamma_n,\delta_n,v_{0,n}),\qquad v:=S(\gamma,\delta,v_0).$$

Let $t>0$. Due to Lemmas 3.2 and 4.1,

$$\|v_n(t,\cdot)-v(t,\cdot)\|_{H^1(\mathbb{R})}=\lim_{\varepsilon\to 0}\|v_{\varepsilon,n}(t,\cdot)-v_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}.$$

Using [2, Theorem 3.1], we have that

$$\|v_{\varepsilon,n}(t,\cdot)-v_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}\leq A(t,\varepsilon)\|v_{0,n}-v_0\|_{H^1(\mathbb{R})}+B(t,\varepsilon)\big(|\gamma_n-\gamma|+|\delta_n-\delta|\big),$$

with

$$A(t,\varepsilon)=\mathcal{O}(e^{T/\varepsilon}),\qquad B(t,\varepsilon)=\mathcal{O}(e^{T/\varepsilon}),\quad t\in[0,T].$$

Now, using the same argument as in [3, Lemma 8.1] we prove the claim. □

Proof of Theorem 1.1. It is direct consequence of Lemmas 4.2 and 4.4. □

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(Mostafa Bendahmane, Giuseppe Maria Coclite, and Kenneth Hvistendahl Karlsen)
Centre of Mathematics for Applications (CMA)
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway

E-mail address: `mostafab@math.uio.no, giusepc@math.uio.no, kennethk@math.uio.no`

URL: `http://folk.uio.no/mostafab/, http://www.math.uio.no/~kennethk/`